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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A direct state space approach to the digital control of multivariable linear systems is presented. Control is provided by minimizing the mean square error between controlled plant outputs and specified desired output trajectories at sampling instants. Linear equations for digital control inputs are solved in constant forward and feedback gain form. Numerical applications to problems in simple model following, digital redesign, and direct digital design are given.

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MULTIVARIABLE LINEAR DIGITAL CONTROL VIA

STATE SPACE OUTPUT MATCHING*

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ABSTRACT

A direct state space approach to the digital control of multivariable linear systems is presented. Control is provided by minimizing the mean square error between controlled plant outputs and specified desired output trajectories at sampling instants. Linear equations for digital control inputs are solved in constant forward and feedback gain form. Numerical applications to problems in simple model following, digital redesign, and direct digital design are given.

KEY WORDS   Digital control Linear feedback systems   Output matching

Digital redesign   Direct digital design

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1. INTRODUCTION.

Recent advances in microcomputer technology have generated renewed interest in the design and implementation of sampled-data control systems, a subject area with an already mature and well-developed theory.[1,2] The introduction of digital flight control systems into modern aircraft, for example, has for the past several years been receiving considerable attention.[3-6] The compactness, flexibility, accuracy, and reliability of flight control computers make them ideally suited for the exacting guidance and control requirements of high speed flight environments. Desired future full-scale integration of the diverse control functions of aircraft subsystems will necessitate efficient utilization of digital processors. Accordingly, this discussion focuses upon a simple, direct, and computationally efficient signal tracking approach to the digital control of multivariable linear systems.

Signal tracking has, of course, provided an effective design criterion for a variety of practical control problems. The popular linear quadractic regulator[7] permits the close tracking of reference variables (set points) which are constant over relatively long periods of time while at the same time constraining the amplitudes of control inputs. The MRA techniques surveyed by Landau[8] have as their common objective the minimization of deviations between reference and controlled system outputs through the adjustment of system parameters and various gain matrices. Kuo, Singh, and Yackel[9,10] have employed output matching at sampling instants to digitalize the control of continuous feedback systems. Mehra[4,5,11] has used output matching, in conjunction with an impulse response plant representation, to drive system outputs to desired set points along constructed nonlinear trajectories.

This paper presents a state space output matching approach to the digital control of continuous multivariable linear systems. The control strategy adopted relies simply upon the sequential generation of digital inputs which force system outputs to closely track prescribed trajectories. The problem formulation is quite general and is very similar in flavor to that of Mehra. It provides, as will be seen, an alternate approach to the problems considered in references 4 and 9. The control technique developed relies upon only the most elementary principles of linear systems theory and hence is characterized by its conceptual and computational simplicity.

2. THE CONTROL PROBLEM.

Consider the sampled-data feedback control, as depicted in Figure 1, of the time-invariant dynamical system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t), \quad x(0) = x_0. \tag{1}$$

Here $T > 0$ is the sampling period, $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^\ell$, $z(t) \in \mathbb{R}^\ell$,

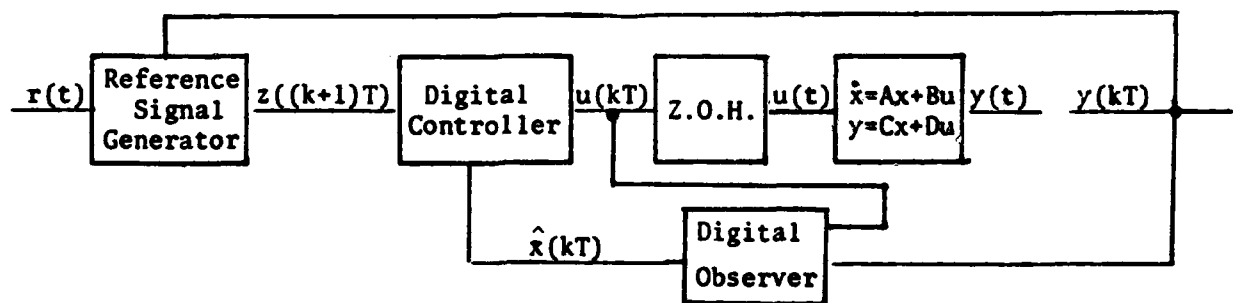Figure 1 - Sampled-data Control System

$r(t) \in \mathbb{R}^p$, and $u(t) \in \mathbb{R}^m$ satisfies

$$u(t) = u(kT), \quad kT \leq t < (k+1)T, \; k = 0,1,2... \qquad (2)$$

The matrices A, B, C, and D are of appropriate dimensions. For later con-
venience, if all states are observable, assume that the observer is omitted
and that x(kT) is fed back to the controller.

The function z(t) represents a desired trajectory that the plant output
y(t) is to track closely. It may be the output of a dynamical system as
predicted by its model driven by r(t), or some other appropriately generated
signal. However z(t) is specified, it is assumed that z((k+1)T) is known
at t = kT (See Figure 1). Close tracking is insured here, as in references
4, 5, 9-11, by determining a control variable sequence u(0), u(T), u(2T),...
that insures close matching of the components $y_i(t)$ and $z_i(t)$ at t = kT,
k = 0,1,2,...

As will be seen, in the absence of control constraints, the solution
of the control problem considered reduces simply to determing optimal con-
stant gain matrices F and G and a sequence of vectors c(kT) for the control
system of Figure 2. This, of course, is the attractive form of a standard
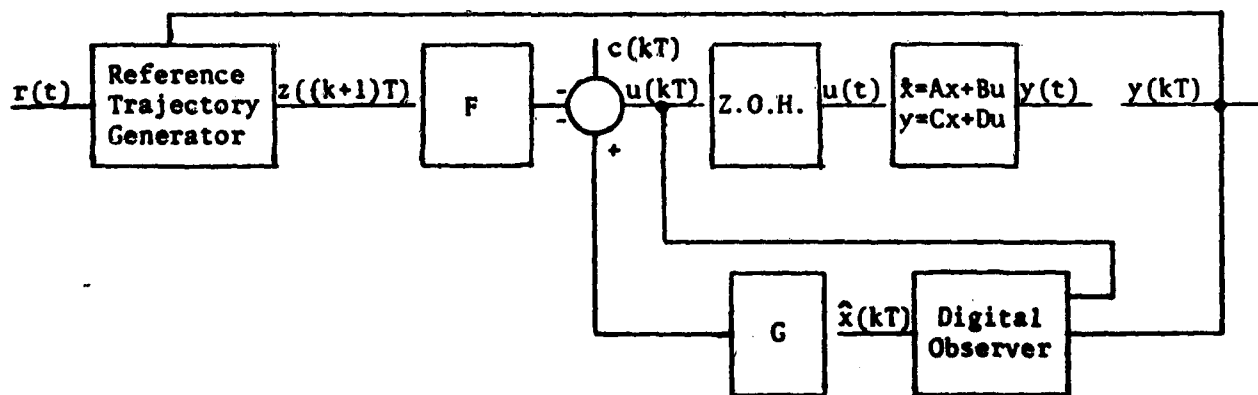linear feedback control system.

Figure 2 - Constant Gain System

3. OPTIMAL CONTROL STRATEGIES.

Since controls u(t) in (2) are piecewise constant, solutions x(t) of (1) satisfy the difference equations

$$x((k+1)T) = \Phi x(kT) + \Theta u(kT), \quad k = 0,1,2,\ldots , \tag{3}$$

where

$$\Phi = e^{AT} \quad \text{and} \quad \Theta = \int_0^T e^{A(T-\tau)} B \, d\tau . \tag{4}$$

Using these equations, a precise formulation of the output matching control problem of section 2 is easily given. Specifically, at each sampling instant $t = kT$, $k \geq 0$, given $y(kT)$ and $z((k+1)T)$, consider the unconstrained minimization of

$$J(u(kT)) = \frac{1}{2} \sum_{i=1}^{\ell} q_i [z_i((k+1)T) - y_i((k+1)T)]^2$$

$$= \frac{1}{2} \sum_{i=1}^{\ell} q_i [z_i((k+1)T) - C_i \Phi x(kT) - (C_i\Theta+D_i) u(kT)]^2 . \tag{5}$$

Here $q_i > 0$, $1 \leq i \leq \ell$, C_i and D_i are the ith rows of C and D respectively, and $x((k+1)T)$ is given by (3). The control u(t) determined through (2) by the variables u(0), u(T), u(2T),... which minimize the functions $J(u(kT))$ sequentially, $k = 0,1,2,\ldots$, will be said to be *optimal*. If the more typical problem of minimizing $\sum_{k\geq 0} J(u(kT))$ is considered, then the strategy considered here clearly may be suboptimal. However, if the plant is sufficiently responsive and T is sufficiently small, the minimization of $J(u(kT))$ should provide desired close tracking of z(t).

Consider the minimization of $J(u(kT))$ when all states are observable and controls are unconstrained (Figure 1 with the observer omitted and $x(kT)$ fed back). In theory, perfect output matching ($J(u(kT)) = 0$ for all k) can be achieved if and only if the equations

$$(C\Theta + D)u(kT) = z((k+1)T) - C\Phi x(kT) \qquad (6)$$

have a solution (cf. Kuo, Singh, and Yackel[9]). If a solution exists and if rank $(C\Theta + D) = r$, then there exists a permutation α of the indices $\{1,2,\ldots,m\}$, with $\alpha(i) < \alpha(i+1)$, and matrices M and N such that

$$\begin{bmatrix} u_{\alpha(1)}(kT) \\ \vdots \\ u_{\alpha(r)}(kT) \end{bmatrix} = M[z((k+1)T) - C\Phi x(kT) - N\begin{bmatrix} u_{\alpha(r+1)}(kT) \\ \vdots \\ u_{\alpha(m)}(kT) \end{bmatrix}] . \qquad (7)$$

This follows from the row reduction of r linearly independent rows of $C\Theta + D$. For $r < m$, N consists of columns $\alpha(r+1)$, $\alpha(r+2)$,..., $\alpha(m)$ of $C\Theta + D$. The values of the variables $u_{\alpha(i)}(kT)$, $r + 1 \leq i \leq m$, may be conveniently assigned arbitrarily. Thus, the solution vector $u(kT)$ may be written

$$u(kT) = F z((k+1)T) - Gx(kT) - c(kT). \qquad (8)$$

The rows $F_{\alpha(i)}$ of F are given by $F_{\alpha(i)} = M_i$ for $1 \leq i \leq r$, and $F_{\alpha(i)} = 0$ otherwise. The matrix G is given by $G = FC\Phi$ and, letting $c'(kT) = FN[u_{\alpha(r+1)}(kT),\ldots,u_{\alpha(m)}(kT)]^T$, the correction vector $c(kT)$ has $c_{\alpha(i)}(kT) = c'_{\alpha(i)}(kT)$, $1 \leq i \leq r$, and $c_{\alpha(i)}(kT) = -u_{\alpha(i)}(kT)$ otherwise (superscript T denotes transpose). If $r = m$, $F = M = (C\Theta + D)^{-1}$ and $c(kT) \equiv 0$.

Importantly, F and G (and effectively the vectors $c(kT)$, since $u_{\alpha(i)}(kT)$, $r + 1 \leq i \leq m$, is arbitrary) are independent of k. The optimal control $u(t)$ therefore has the constant gain configuration indicated in Figure 2.

More generally, if equations (6) have no solution, $J(u(kT))$ may be minimized directly. A necessary and sufficient condition for its uncon-strained minimization, since it is convex in $u(kT)$, is $\nabla J(u(kT)) = 0$. This condition reduces simply to

$$(C\Theta + D)^T Q(C\Theta + D)u(kT) = (C\Theta + D)^T Q[z((k+1)T) - C\Phi x(kT)] \qquad (9)$$

where $Q = (q_{ij})$ with $q_{ij} = \delta_{ij}q_i$ ($\delta_{ij} = 1$, $i = j$, $\delta_{ij} = 0$ otherwise). Solutions of (9) can be written in the form (8). If equations (6) are solvable, then, in view of the convexity of $J(u(kT))$, the solution spaces of (6) and (9) coincide. Obviously then, attention may be restricted exclusively to equations (9).

When unobservable states are present, an asymptotically stable digital observer $\hat{x}$ defined by

$$\hat{x}((k+1)T) = \Phi \; \hat{x}(kT) + \Theta u(kT) + \hat{G}[y(kT) - C\hat{x}(kT)], \quad k=0,1,2,\ldots, \qquad (10)$$

is employed. In this case, y may be controlled most directly by minimizing $J(u(kT))$ under the assumption of complete observability, and then substituting the estimate $\hat{x}(kT)$ for $x(kT)$ in (8). Alternatively, the observer output $\hat{y}((k+1)T) = C\hat{x}((k+1)T)$ may be substituted for $y((k+1)T)$ in (5), with $\hat{x}((k+1)T)$ given by (10). The altered $J(u(kT))$ may then be minimized as before. The optimal control again assumes the basic form (8), but with $\hat{x}(kT)$ replacing $x(kT)$ and an additional term $Hy(kT)$ present due to the dependence of $\hat{y}((k+1)T)$ upon $y(kT)$. Obviously, the effectiveness of either of these control schemes depends upon how rapidly $\hat{x}(kT)$ converges to $x(kT)$, as k increases, for given initial conditions $x(0)$ and $\hat{x}(0)$.

Remarks

3.1 Exact output matching can in general be achieved only when $\ell < m$, that

is, only when there are at least as many control variable degrees of freedom as there are controlled variables. In practice, if $\ell = m$, one expects (6) to possess a unique solution, implying that $C\Theta + D$ is invertible, and hence that $c(kT) \equiv 0$ in (8). If $\ell < m$, one expects that the values of $m - \ell$ control variables may be assigned freely, so that, in general, $c(kT) \neq 0$. Similarly, multiple solutions in (9) are expected when $\ell < m$, and a unique solution when $\ell \geq m$. The later implies that the matrix $(C\Theta + D)^T Q(C\Theta + D)$ is invertible, clearly a desirable property.

3.2 Because $J(u(kT))$ is continuous, it will always possess a constrained minimum if constraints $a_i \leq u_i(kT) \leq b_i$ are imposed. This minimum can be determined numerically on each sampling interval using standard nonlinear programming algorithms. As in regulator problems, constraints may also be imposed through the minimization of $\tilde{J}(u(kT)) = J(u(kT)) + u(kT)^T R\, u(kT)$, $R > 0$. Equating $\nabla\tilde{J}(u(kT))$ to zero, a system of linear equations for the components of $u(kT)$ results with solutions again of form (8). Alternatively, constraints may be effectively imposed in some cases, as when transferring outputs to set points, by selecting only those paths $z(t)$ which require bounded control energy for close tracking.

3.3 The derivations of equations (6) and (9) are based simply upon the difference equations (3) which solutions of (1) with controls (2) satisfy. These equations are valid more generally if $A = A(t)$ and $B = B(t)$ are matrices of measurable functions which are bounded on finite intervals (see Coddington and Levinson[12]). In this case, however, $\Phi = \Phi(k)$ and $\Theta = \Theta(k)$ depend upon k. Consequently, the coefficient matrices in (6) and (9) and hence the gains $F = F(k)$ and $G = G(k)$ in (8) also depend upon k. The on-line implementation of the control scheme discussed here may then present serious computational difficulties. This is particularly true if T is very small,

since the matrices of integrals $\Phi(k)$ and $\Theta(k)$ must be computed and a system of linear equations (6) or (9) must be solved at each sampling instant $t = kT$. Moreover, if unobservable states are present, the additional problem of choosing gains $\hat{G} = \hat{G}(k)$ so that $\hat{x}$ in (10) is asymptotically stable must be addressed.

4.   NUMERICAL EXAMPLES.

Example 1.   *A model following problem.*

Let a one dimensional plant be described by the state and output equations

$$\dot{x}(t) = -x(t) + u(t)$$

$$y(t) = x(t) \ .$$

(11)

Let the equations of a second reference system be

$$\dot{q}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix} q(t) + \begin{bmatrix} 0 \\ 2 \end{bmatrix} r(t) \ . \tag{12}$$

Consider the problem of finding a control $u(t)$ of form (2) that forces $y(t)$ to match $z(t) = q_1(t)$ closely at sampling instants.

In (6), $C\Theta + D = \int_0^T e^{-(T-\tau)}d\tau = 1 - e^{-T}$ so that exact output matching can be achieved using $u(t)$ determined by

$$u(kT) = Fz((k+1)T) - Gx(kT) \tag{13}$$

with

$$F = \frac{1}{1 - e^{-T}} \quad \text{and} \quad G = \frac{e^{-T}}{1 - e^{-T}} \ . \tag{14}$$

Figure 3 shows z(t) and the piecewise exponential function y(t) as controlled by (13) and (14). Here T = 1, r(t) is a unit step function, and x(0) = q(0) = 0.

If only $q_2(t)$ were observable in (12), it would be necessary to equip q(t) with an observer

$$\hat{q}((k+1)T) = \Phi^q q(kT) + \Theta^q r(kT) + \hat{G}^q [q_2(kT) - \hat{q}_2(kT)] ,$$

where

$$\Phi^q = \begin{bmatrix} .371074 & .444476 \\ -.888951 & -.073402 \end{bmatrix}, \Theta^q = \begin{bmatrix} .628926 \\ .888952 \end{bmatrix}, \text{ and } \hat{G}^q = \begin{bmatrix} .289579 \\ .297672 \end{bmatrix} .$$

(See Kuo[1] for digital observer design.) Letting $z(t) = \hat{q}_1(t)$, y(t) could then be matched to $\hat{q}_1(t)$ using (13) and (14). A control simulation in this situation is pictured in Figure 4. T, r(t), and x(0) and q(0) are as above. The simulation begins with $\hat{q}(0) = 0$. At t = 3 it is assumed that the actual state q(t), evolving according to (12), is subjected to a disturbance which ..esults in $q_1(3) = q_1(3^-) + .2$ and $q_2(3) = q_2(3^-) + .3$. Figure 4 illustrates the convergence of y(t) to $q_1(t)$ as $\hat{q}(kT)$ converges to q(kT) from $\hat{q}(3) = q(3^-)$.

Since J(u(kT)) is convex in a single variable, it is easily minimized subject to the constraint $a \leq u(kT) \leq b$. Clearly, if $\nabla J(\alpha) = J'(\alpha) = 0$, to minimize J(u(kT), choose $u(kT) = \alpha$ if $a \leq \alpha \leq b$, and choose u(kT) = a if $\alpha < a$ and u(kT) = b if $\alpha > b$. Figure 5 shows the example of Figure 4 simulated subject to the constraint $0 \leq u(kT) \leq 1$.

Example 2. *A digital redesign problem.*

Kuo, Singh, and Yackel[9] consider the problem of converting a continuous-data control system into a sampled-data system while preserving the

performance characteristics of the original system. Sample and hold devices are inserted into the continuous system and its input and feedback gains are modified. The modifications insure that linear combinations of the plant states of the resulting sampled-data system match the same combinations of states of the original system *exactly* at all sampling instants.

The output matching technique of section 3 may also be applied to the digital redesign problem. To illustrate this, consider the simplified one-axis *Skylab* satellite control system of reference 9 with plant equations

$$\dot{q}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} q(t) + \begin{bmatrix} 0 \\ \dfrac{1}{970,741} \end{bmatrix} u(t) \ , \ q(0) = 0 \tag{15}$$

and control

$$u(t) = E(0) \ r(t) - G(0) \ x(t). \tag{16}$$

The scalar input $r(t)$ is taken to be a unit step function, and the input and feedback gains $E(0)$ and $G(0)$ are

$$E(0) = 11,800 \ \text{ and } \ G(0) = [11,800 \ \ 151,800] \ .$$

Given a 1 x 2 matrix C, formulas for modified gains $E(T)$ and $G(T)$ are derived in reference 9 with the property that if $q^d(t)$ is the solution of (15) with $u(t)$ given by

$$u(t) = E(T) \ r(kT) - G(T) \ x(kT), \ kT \leq t < (k+1)T, \ k = 0,1,2,\ldots,$$

then $Cq^d(t)$ and $Cq(t)$ match exactly at all sampling instants.

Let $x(t)$ represent the solution of (15) if $u(t)$ is of form (2), while $q(t)$ is *the* solution with control (16). The difference equations (3) satisfied by $x(t)$ are

$$x((k+1)T) = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} x(kT) + \begin{bmatrix} \dfrac{T^2}{2,941,482} \\ \dfrac{T}{970,741} \end{bmatrix} u(kT), \quad x(0) = 0, \quad k = 0,1,2,\ldots \quad (17)$$

Given a 1 x 2 matrix C, in the framework of section 3, the redesign goal is to determine a control (2) which insures the close matching of $y(t) = Cx(t)$ and $z(t) = Cq(t)$. By (6), if the scalar $C\Theta \neq 0$, $y(t)$ and $z(t)$ may be matched exactly at sampling instants using (13) with gains

$$F = (C\Theta)^{-1} \quad \text{and} \quad G = (C\Theta)^{-1}C\Phi . \tag{18}$$

This is the same result derived in reference 9 for the matching of $Cq^d(t)$ and $Cq(t)$.

Reference 9 discusses in detail the cases $C = [1 \ 0]$ and $C = [0 \ 1]$, with $T = 2$. The corresponding gains (18) for these cases are

$$F = \begin{cases} \dfrac{1}{\Theta_1} & C = [1 \ 0] \\ \dfrac{1}{\Theta_2} & C = [0 \ 1] \end{cases} \quad \text{and} \quad G = \begin{cases} [\dfrac{1}{\Theta_1} \ \dfrac{T}{\Theta_1}] & C = [1 \ 0] \\ [0 \ \dfrac{1}{\Theta_2}] & C = [0 \ 1] , \end{cases} \tag{19}$$

where $\Theta = [\Theta_1 \ \Theta_2]^T$. Since both redesign strategies provide exact output matching, the simulation results for the above choices of C are identical for each method. Figures 4 - 13, reference 9 (or Figures 6.19 - 6.24, Kuo[1]) illustrate these results. Although both states of (15) are matched well for each choice of C, it is seen that better simultaneous matching of states occurs when $C = [0 \ 1]$.

Kuo, Singh, and Yackel consider the digitalization of the control of a continuous plant (1) in the case $\ell = m$. (C here is the matrix H of reference 9.) More generally, the output matching technique presented here applies to the digital redesign problem when ℓ and m are arbitrary. As

indicated in section 3, when D = 0, a practical condition permitting exact output matching is the existence of $(C\Theta)^{-1}$. This is the same condition given in reference 9 for the case $\ell = m$.

To illustrate the more general problem, letting C = I, the identity matrix, consider the *Skylab* satellite with z(t) = Cq(t) = q(t). For control, minimize the weighted sum

$$J(u(kT)) = \frac{1}{2}\,\overline{q}_1[q_1((k+1)T) - x_1((k+1)T)]^2 + \frac{1}{2}\,\overline{q}_2[q_2((k+1)T) - x_2((k+1)T)]^2.$$

From (9), gains F and G for the optimal control (13) are given by

$$F = \alpha^{-1}(C\Theta)^T Q \quad \text{and} \quad G = \alpha^{-1}(C\Theta)^T QC\Phi \tag{20}$$

where $Q = (q_{ij})$ has $q_{ij} = \delta_{ij}\,\overline{q}_i$ and the scaler $\alpha = (C\Theta)^T Q(C\Theta)$ is assumed nonzero. Figures 6, 7, and 8 illustrate simulations of q(t) and x(t) as controlled through (13) and (20) with T = 2, $\overline{q}_1 = 2$, and $\widetilde{q}_2 = 1$. As can be seen, close matching of both states simultaneously is provided. (The results are effectively unchanged when $\overline{q}_1 = 1$ and $\overline{q}_2 = 2$, or when $\overline{q}_1 = \overline{q}_2 = 1$.) In fact, when the actual data is analyzed, slightly better simultaneous matching is provided by C = I than by either C = [1 0] or C = [0 1].

To illustrate the incorporation of the digital observer of Figure 2 into the control scheme, consider the problem of matching $q_2(t)$ and $x_2(t)$ when only $x_1(t)$ is observable. Equip x(t) with an observer (10) with gain $\hat{G} = [2 \quad 1/T]^T$. For control, use (13) and (19) with C = [0 1], but with $\hat{x}(kT)$ replacing x(kT) in (13). A control simulation under these conditions is illustrated in Figures 9, 10, and 11. Here again T = 2 and q(0) = x(0) = 0. It is assumed that $\hat{x}_1(0) = 0$, but that the initial estimate $\hat{x}_2(0) = .1$ is in error. Since $\hat{x}(kT)$ effectively converges to x(kT) after 2 sampling periods, x(t) and q(t) are closely matched for $t \geq 6$.

Example 3. *A flight control example.*

In Reference 4, Mehra, (et al) applies the Model Algorithmic Control (MAC) technique to the attitude control of a hypothetical missile. The three axis model, with independent pitch axis and coupled roll-yaw dynamics, is described by the state equations

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} Z_w & 1 & 0 & 0 & 0 & 0 \\ M_\alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & Y_\beta & \alpha_1 & -1 & \frac{g}{v}\cos\phi_1 \\ 0 & 0 & L_\beta & -\frac{1}{2}L_{\delta p} & 0 & -\frac{1}{2}L_{\delta p} \\ 0 & 0 & N_\beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ M_{\delta q} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & L_{\delta p} & 0 \\ 0 & 0 & N_{\delta r} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad (21)
$$

The output, state, and control variables are

$y_1 = x_1$ = angle of attack

x_2 = perturbed pitch rate $\qquad u_1$ = elevator angle

$y_2 = x_3$ = sideslip angle $\qquad u_2$ = aileron angle

x_4 = perturbed roll rate $\qquad u_3$ = rudder angle

x_5 = perturbed yaw rate

$y_3 = x_6$ = roll angle

Angles are measured in degrees.

With the missile flying at Mach 2 at 20,000 ft. and weighing 239.5 lb., the parameters in (21) have values

$$Z_w = -1.4868 \qquad N_\beta = 290.48 \qquad v = 2174.21$$

$$M_\alpha = -149.93 \qquad \alpha_1 = .15708 \qquad \phi_1 = 0.0$$

$$Y_\beta = -.91237 \qquad L_{\delta p} = 8770.6 \qquad M_{\delta q} = -281.11$$

$$L_\beta = -1559.2 \qquad g = 32.0 \qquad N_{\delta r} = 281.11$$

Under these conditions, consider the problem of transferring the outputs $y_1(t)$, $y_2(t)$ and $y_3(t)$ smoothly from given initial values to constant set points c_1, c_2, and c_3 along desired trajectories $z_1(t)$, $z_2(t)$, and $z_3(t)$ respectively. Letting C be defined so that $y(t) = Cx(t)$, the matrix $C\Theta$ is invertible. By (6) then, exact matching of $y(kT)$ and $z(kT)$, $k = 1,2,\ldots,$ may be achieved using $u(t)$ defined through (13) with

$$F = \begin{bmatrix} -.846826 & 0.0 & 0.0 \\ 0.0 & .067512 & 5.269492 \\ 0.0 & -.938841 & .080139 \end{bmatrix} \qquad (22)$$

and

$$G = \begin{bmatrix} -.223562 & -.060463 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & -.107461 & .001085 & .003071 & 4.769538 \\ 0.0 & 0.0 & .014207 & .000023 & .052151 & .080906 \end{bmatrix} \qquad (23)$$

As a particular example, let $z_i(t)$ be defined at sampling instants by

$$z_i(kT) = \alpha y_i((k-1)T) + (1-\alpha)c_i, \quad i = 1,2,3, \ k = 1,2,\ldots, \qquad (24)$$

for constant α, $0 < \alpha < 1$. (The output matching technique requires only that $z(t)$ be specified at $t = kT$.) Figures 12 and 13 show a simulation of (21) as controlled through (13), (22), and (23) with desired trajectories (24) and set points $c_1 = 15°$, $c_2 = 10°$, and $c_3 = 0°$. Here $x(0) = 0$, $T = .1$, $\alpha = .5$, and control action begins at $t = .5$ (i.e., $z(kT) = 0$, and hence $u(kT) = 0$,

$0 \leq k \leq 5$, and $z(kT)$ is given by (24) for $k \geq 6$).

The rapid oscillations of $u(t)$ and $y(t)$ every .1 seconds apparent in Figures 12 and 13 are clearly undesirable. Smoother control of (21) can be realized if more than one future value of $y(t)$ is considered in each control computation. As an example, suppose that at $t = kT$, $u(t)$ is to be held constant at value $u(kT)$ for *two* sampling periods. Consider the minimization of the two step ahead matching criterion

$$J(u(kT)) = \frac{1}{2} \sum_{i=1}^{3} [z_i^1((k+1)T) - y_i((k+1)T)]^2 + \frac{1}{2} \sum_{i=1}^{3} [z_i^2((k+1)T) - y_i((k+2)T)]^2 \quad (25)$$

where for $j = 1,2$, $z_i^j((k+1)T) = z_i((k+1)T)$ as defined through (24) with $\alpha = \alpha_j$, $\alpha_1 > \alpha_2$. Since

$$y_i((k+j)T) = Ce^{A(jT)}x(kT) + C\int_0^{jT} e^{A(jT-\tau)}Bu(kT), \quad j = 1,2,$$

$J(u(kT))$ may be minimized by solving the linear equations $\nabla J(u(kT)) = 0$. The solution of these equations may be placed in the form

$$u(kT) = F^1 z^1((k+1)T) + F^2 z^2((k+1)T) - Gx(kT) \quad (26)$$

for appropriate gains F^1, F^2, and G. Simulations of the control of (21) using controls $u(t)$ determined sequentially through (2) by the $u(kT)$ in (26) (i.e., $u(kT)$ is *applied* on only *one* sampling interval) reveal a marked smoothing of $u(t)$ and $y(t)$. ((25) effectively imposes rate constraints upon the components of $y(t)$ as well as positional constraints.) The simulation with $\alpha_1 = .75$, $\alpha_2 = .5$, and $x(0)$, T, and the c_i as previously specified is pictured in Figures 14 and 15. As can be seen, the components of $u(t)$ and $y(t)$ converge quickly and smoothly to steady state values. The gains F^1, F^2, and G in (26) for this example are

$$F^1 = \begin{bmatrix} -.110352 & 0.0 & 0.0 \\ 0.0 & .127214 & 1.135150 \\ 0.0 & -.239291 & -.002336 \end{bmatrix}, \quad F^2 = \begin{bmatrix} -.285081 & 0.0 & 0.0 \\ 0.0 & -.175194 & 2.164551 \\ 0.0 & -.408877 & .004776 \end{bmatrix}, \quad (27)$$

and,

$$G = \begin{bmatrix} -.104394 & -.028234 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & -.022853 & .000639 & -.000910 & 2.799690 \\ 0.0 & 0.0 & .363652 & .000015 & .007557 & .002551 \end{bmatrix}. \quad (28)$$

R. K. Mehra and others (see references 4, 5, 11, and 13) have employed output matching extensively in the control of linear plants with impulse response representation

$$y_i((k+1)T) = \sum_{j=1}^{\ell} \sum_{n=1}^{N} a(n)_{ij} u_j((k+1-n)T), \quad i = 1,2,\ldots,\ell \ . \quad (29)$$

Letting $z_i(kT)$, $k = 1,2,\ldots$, be a discrete exponential path to the set point c_i, the basic impulse response technique matches $y_i((k+1)T)$ to $z_i((k+1)T)$ using

$$u(kT) = K^{-1}v(k), \quad v_i(k) = z_i((k+1)T) - \sum_{j=1}^{\ell} \sum_{n=2}^{N} a(n)_{ij} u_j((k+1-n)T) \quad (30)$$

where $K = (k_{ij})$ has $k_{ij} = a(1)_{ij}$. Figures 33 a, b, reference 4, show a simulation of the control of (21) using this method with $z_i(kT)$ $= e^{-1}y((k-1)T) + (1-e^{-1})c_i$, $c_1 = 10°$, $c_2 = 15°$, $c_3 = 0°$. The state space model (21) was used to generate the impulse responses $a(n)_{ij}$.

Figures 33 a,b and Figures 12 and 13 here demonstrate essentially equivalent control effectiveness since the desired output trajectories are

similar for both examples and each method provides exact output matching. (The rapid output oscillations of Figures 12 and 13 are absent in Figures 33 a,b. Evidently, linear interpolation has been used to approximate $y(t)$ between the data points generated by (29) and (30). Figures 12 and 13 were constructed using 20 data points in each sampling interval, thus revealing the intersample behavior of the continuous plant (21).) From a computational point of view, however, the state space approach offers the following advantages. Each control computation in (13) with gains (23) requires two matrix-vector multiplications and a vector subtraction using the stored gains (system constants) F and G. Each computation in (30) requires a series of scalar multiplications and additions (to determine $v(k)$), followed by a matrix-vector multiplication, using the system constants $a(n)_{ij}$ and K^{-1} and past control actions $u_j((k+1-n)T)$, $i,j = 1,2,3$, $n = 2,3,\ldots,N$. Table 1 compares the computational requirements of the two control methods.

Methods	System Constants	Algebraic Operations/u(kT)
State Space	$m(\ell + n) = 27(15)$	$m(2\ell + 2n - 1) = 51(27)$
Impulse Response	$(N-1)\ell^2 + m^2 = 450(250)$	$\ell^2(2N-3) + \ell + m(2\ell-1) = 891(495)$

Table 1 - Missile Control Computational Requirements.

(Since the value of N employed in reference 4 is not specified, it is assumed that N = 50 as in reference 11.) Due to the decoupling of states x_1 and x_2 from $x_3 - x_6$ in (21), the matrices F and G in (22) and (23) and F^1, F^2, and G in (27) and (28), and correspondingly the matrix K^{-1} in (26), are naturally partitioned. Taking this into consideration, the actual computational requirements, given in parentheses in Table 1, are the same as the total requirements for two and four-dimensional problems considered separately.

(E.g., 3(3+6) = 27 while 1(1+2) + 2(2+4) = 15.) As can be seen, the state space approach requires significantly fewer parameters to define control with correspondingly fewer computations per control calculation. (Given arbitrary plant equations (1) with ℓ = m and corresponding impulse response representation (29), the formulas in Figure 16 remain in general valid. It is interesting to note that $m^2(2m + 2n - 1) < m^2(2N - 3) + m + m(2m-1)$ whenever $m > \frac{2n-1}{2N-3}$. For N = 50, this implies that the state space technique requires fewer operations per control computation roughly as long as $m > \frac{n}{50}$, that is, as long as there is at least one output for every 50 states.)

As a final comparison, in order to produce control smoothing, reference 4 introduces matching at multiple sampling instants through control blocking. A gradient search algorithm is applied to the nonlinear optimization problem which results. Simulation results using this approach are shown in Figures 34, 37, and 38, reference 4. These figures show smoothing comparable to that of Figures 14 and 15. The minimization of (25) through (26), however, yields improved system performance without introducing additional computational complexity.

V. CONCLUSIONS

A simple, direct, and quite general output matching approach to multivariable linear digital control has been presented. Digital inputs which force plant outputs to closely track desired reference trajectories are generated sequentially by controls with constant forward and feedback gain configuration. Optimal gains are determined using only elementary results from linear algebra and linear systems theory. As a result, the control technique discussed is characterized by its conceptual and practical simplicity.

Applications of output matching to problems in model following, digital redesign, and direct digital design have demonstrated its flexibility, ease of application, and effectiveness. Control effectiveness, of course, depends upon sample rate and desired output trajectory selection. An intersample smoothing strategy promises to provide improved performance without increased sampling rate or additional computational expense. Since each linear plant possesses its own specific dynamical characteristics, it is felt that in application, only experimentation will ultimately reveal those best choices of sampling rate and reference trajectories for given standards of acceptable system performance.

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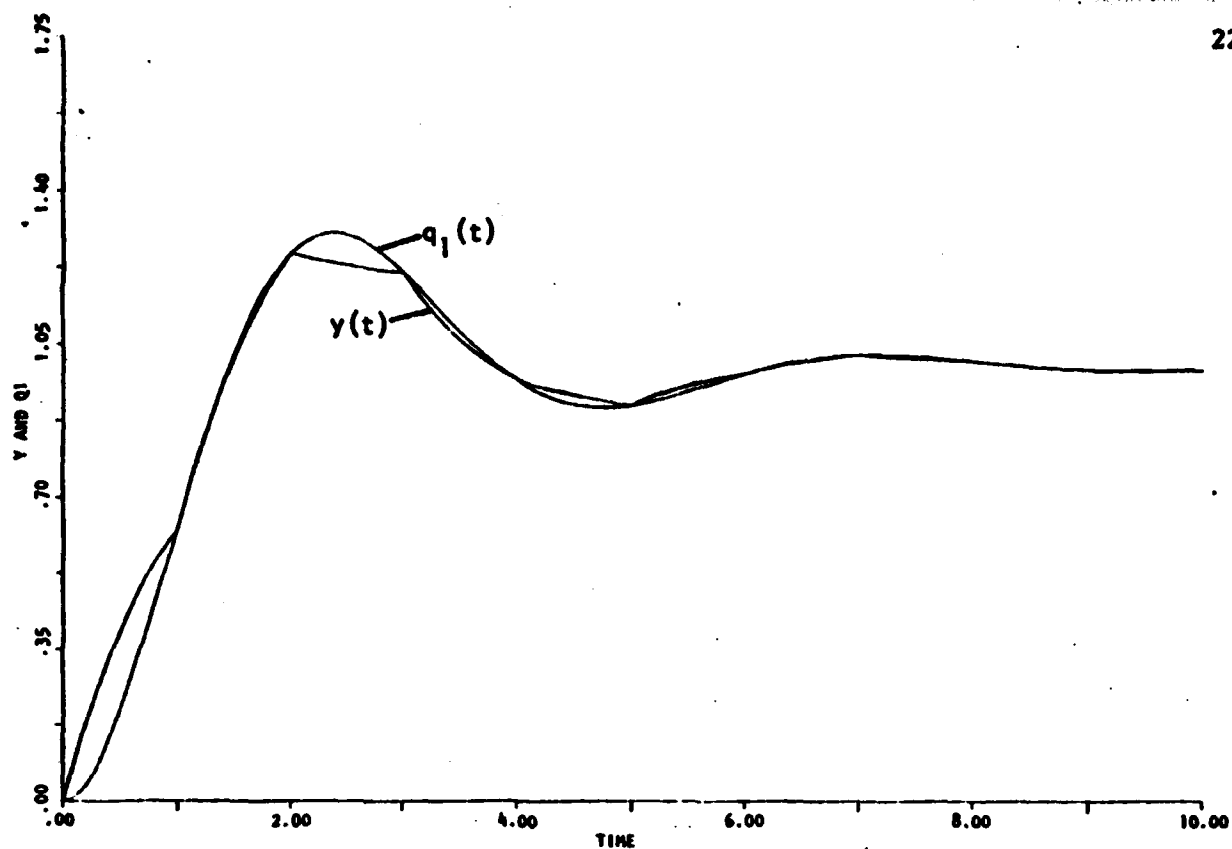
Figure 3. $y(t)$ and $q_1(t)$, Example 1.
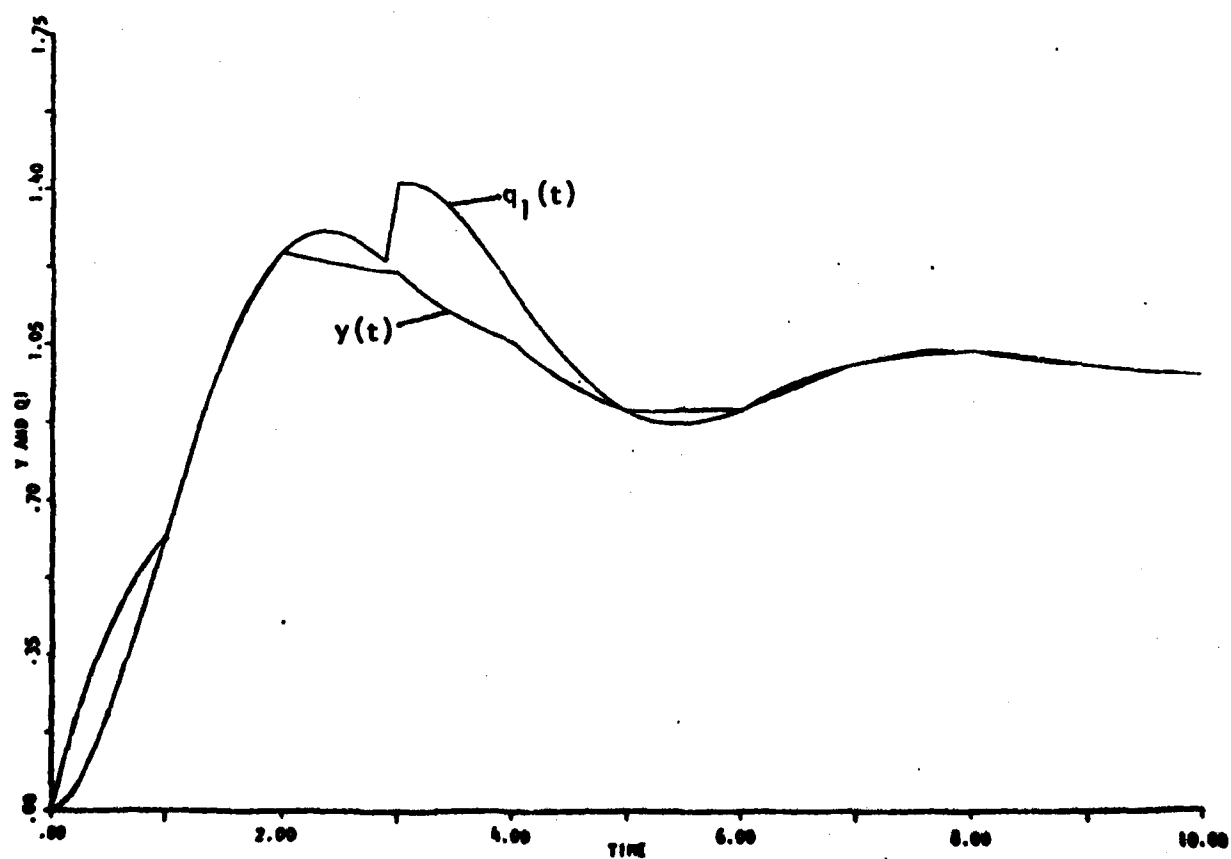
Figure 4. $y(t)$ and $q_1(t)$, Example 1; $q_1(t)$ unobservable.
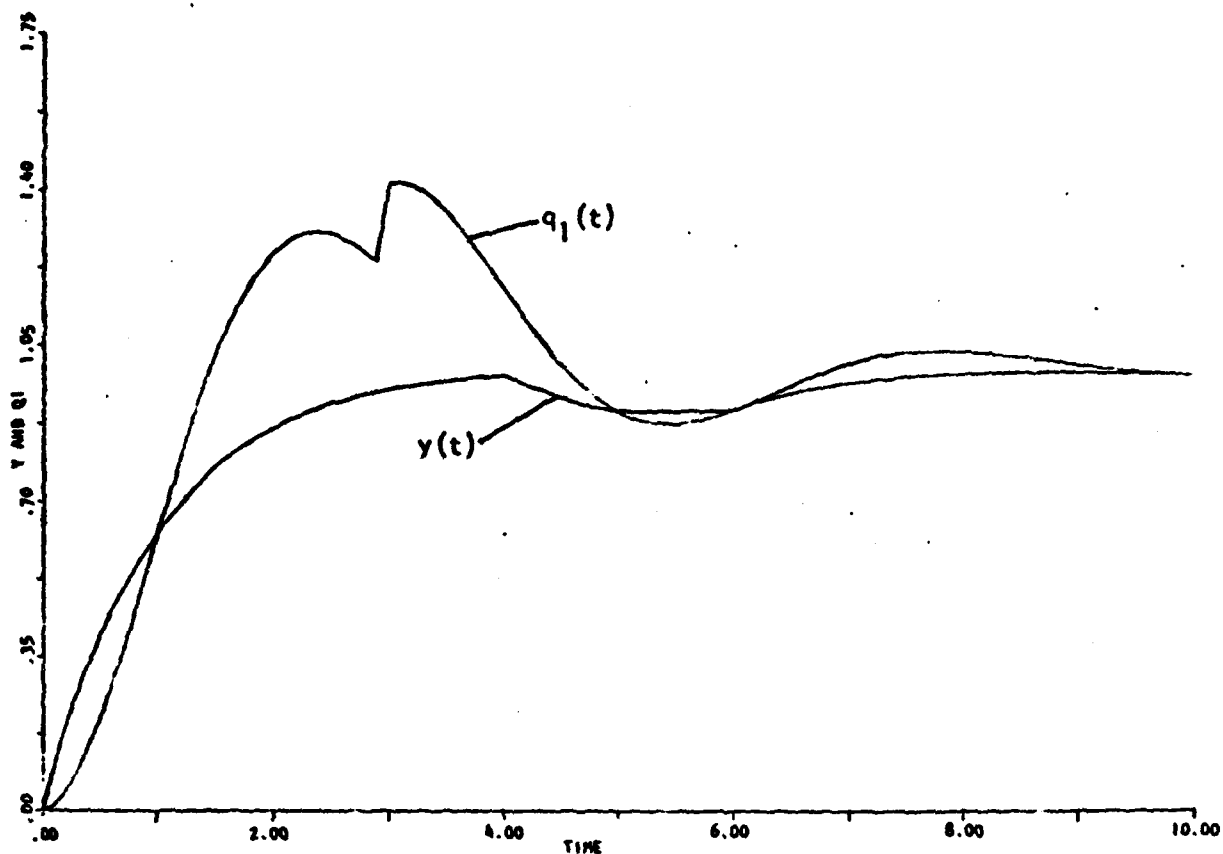
Figure 5. $y(t)$ and $q_1(t)$, Example 1; $q_1(t)$ unobservable, $0 \leq u(t) \leq 1$.
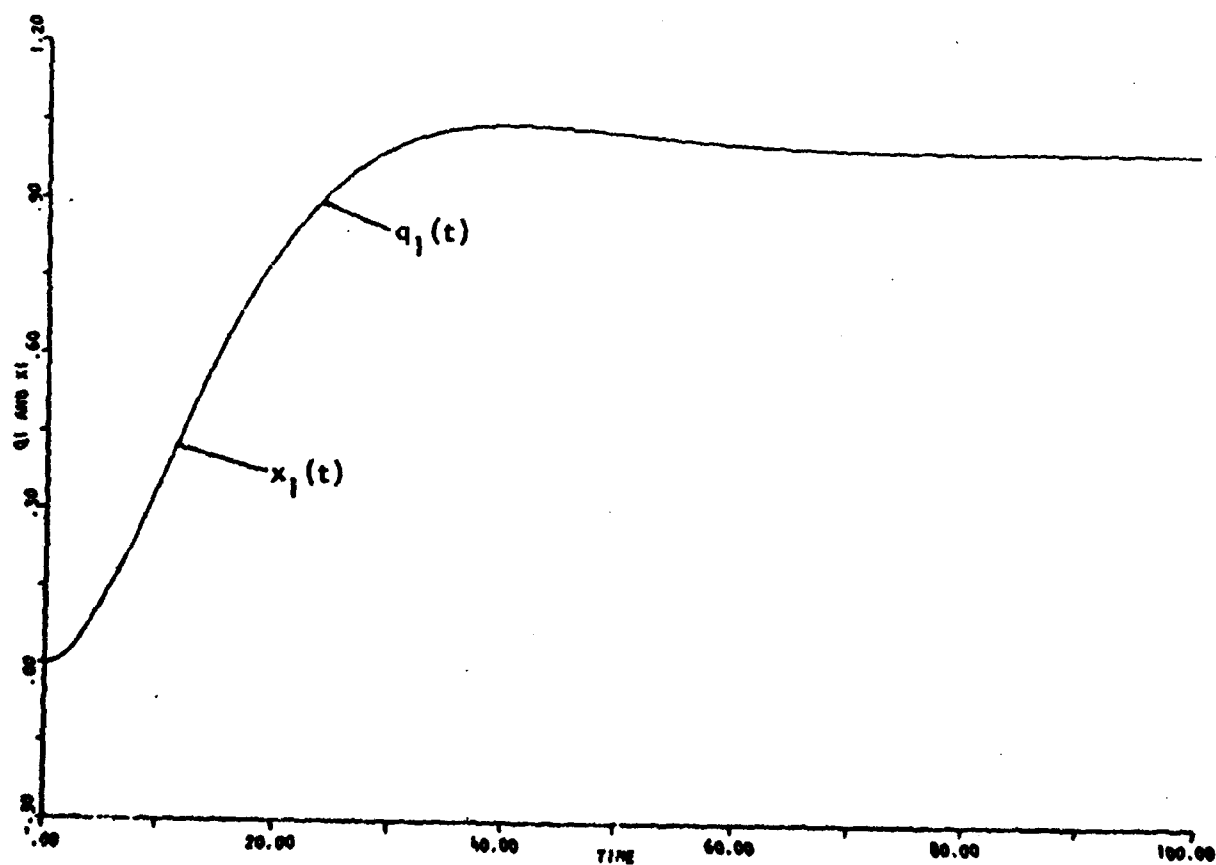
Figure 6. $q_1(t)$ and $x_1(t)$, Example 2; average matching of states, $\bar{q}_1 = 2$, $\bar{q}_2 = 1$.
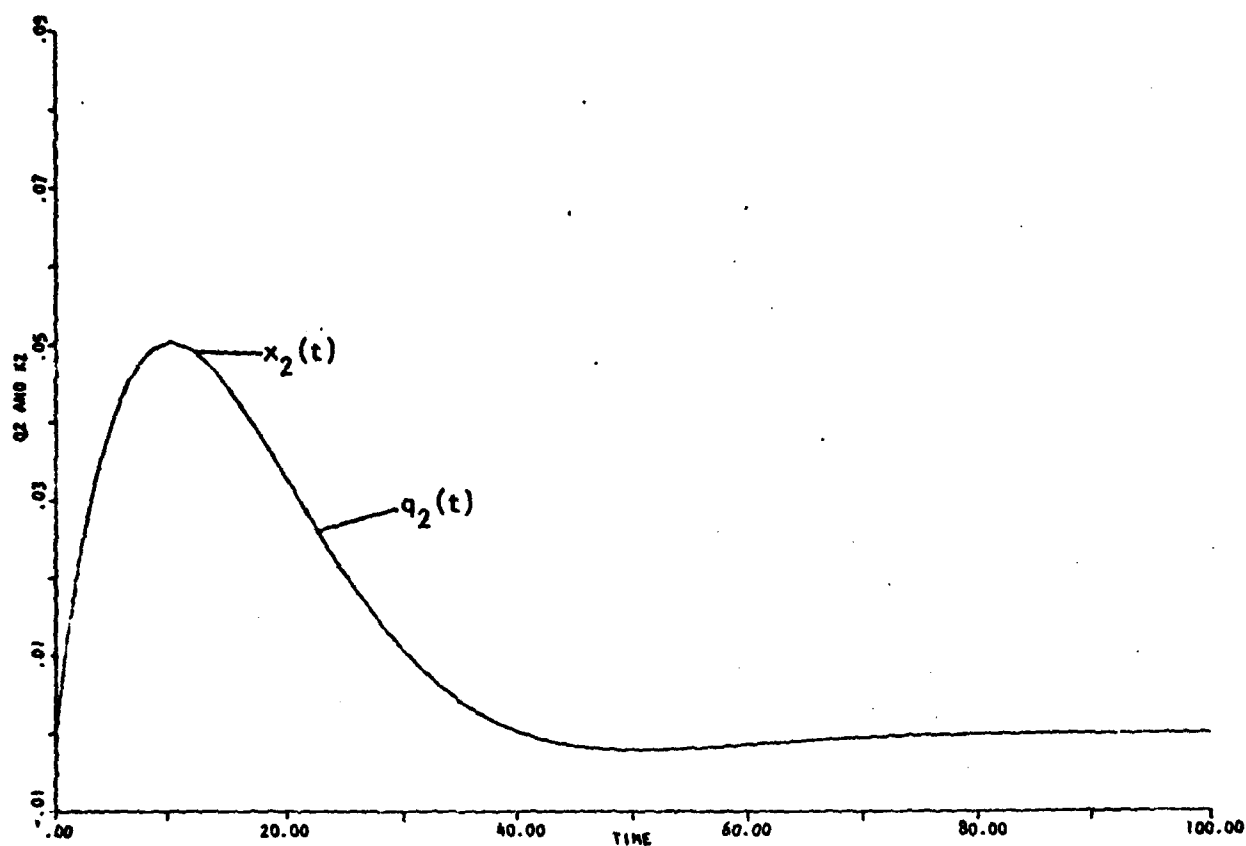
Figure 7. $q_2(t)$ and $x_2(t)$, Example 2; average matching of states, $\overline{q}_1 = 2$, $\overline{q}_2 = 1$.
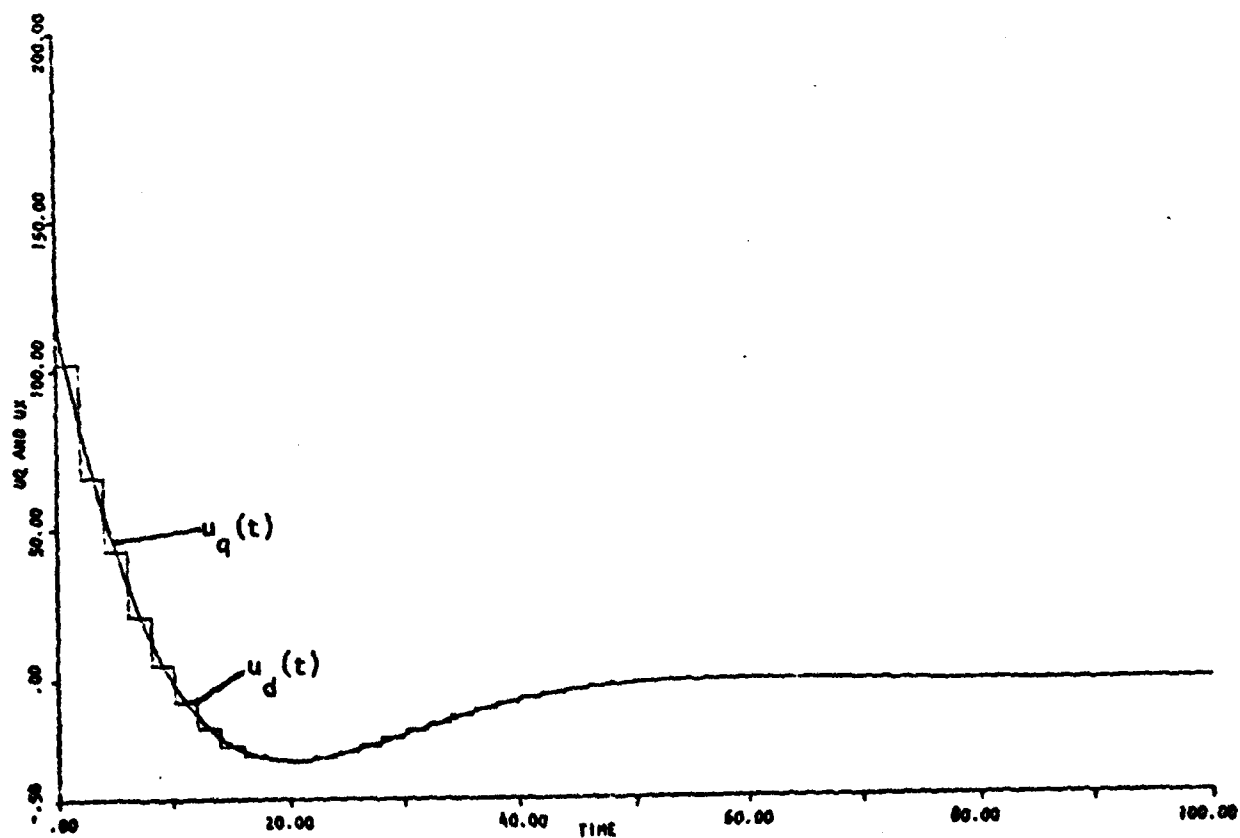
Figure 8.   Control functions, Example 2; average matching of states, $\overline{q}_1 = 2$, $\overline{q}_2 = 1$.
Continuous control $u_q(t)$ given by (16), digital control $u_d(t)$ defined through (13) and (20
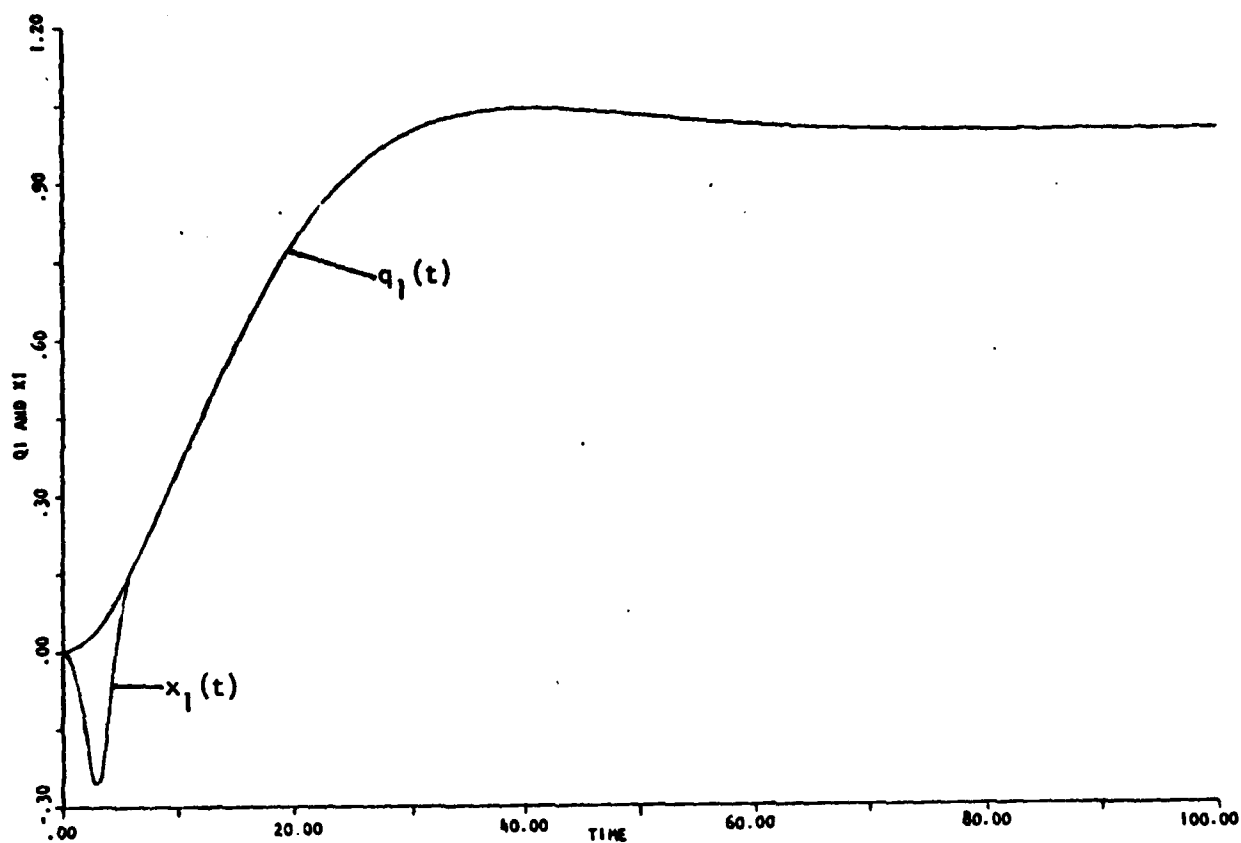
Figure 9. $q_1(t)$ and $x_1(t)$, Example 2; $q_2(t)$ and $x_2(t)$ matched, $x_2(t)$ unobservable.
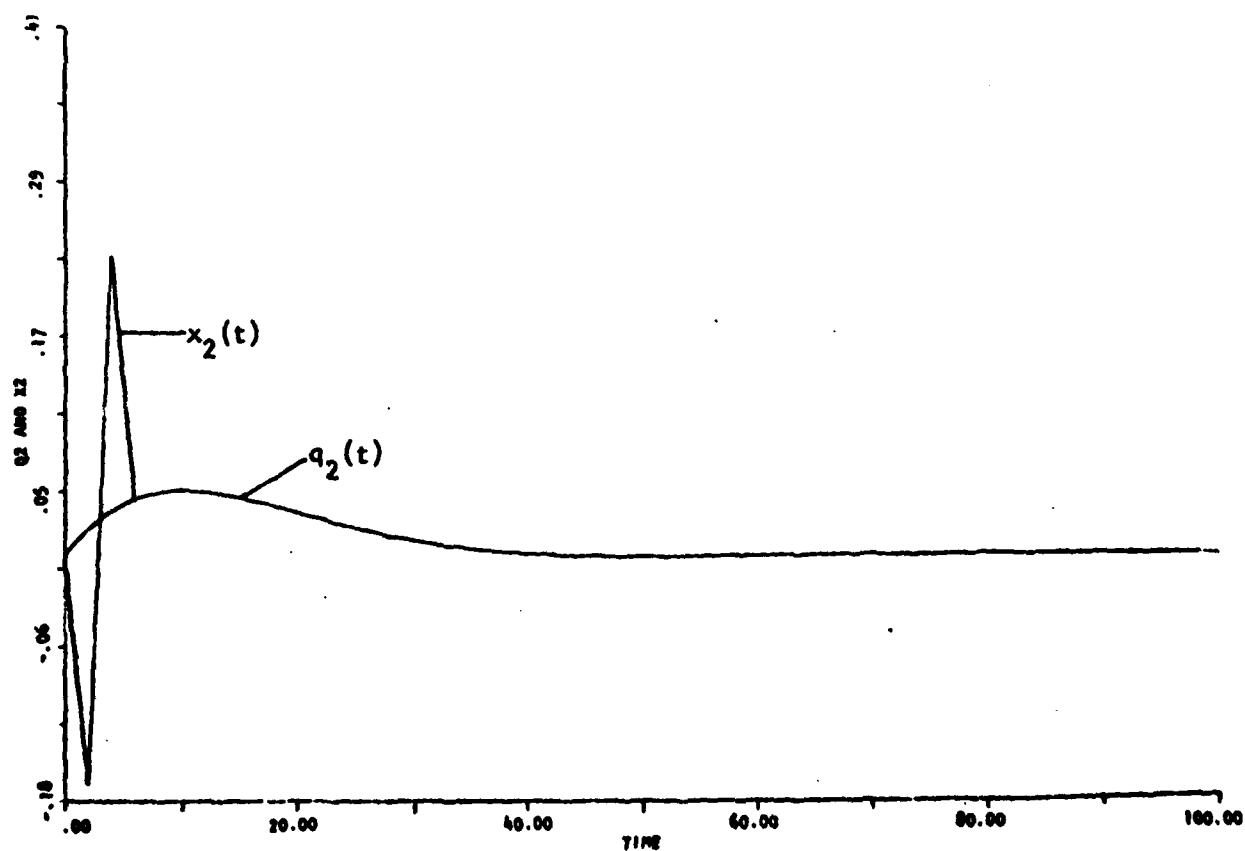
Figure 10. $q_2(t)$ and $x_2(t)$, Example 2; $q_2(t)$ and $x_2(t)$ matched, $x_2(t)$ unobservable.
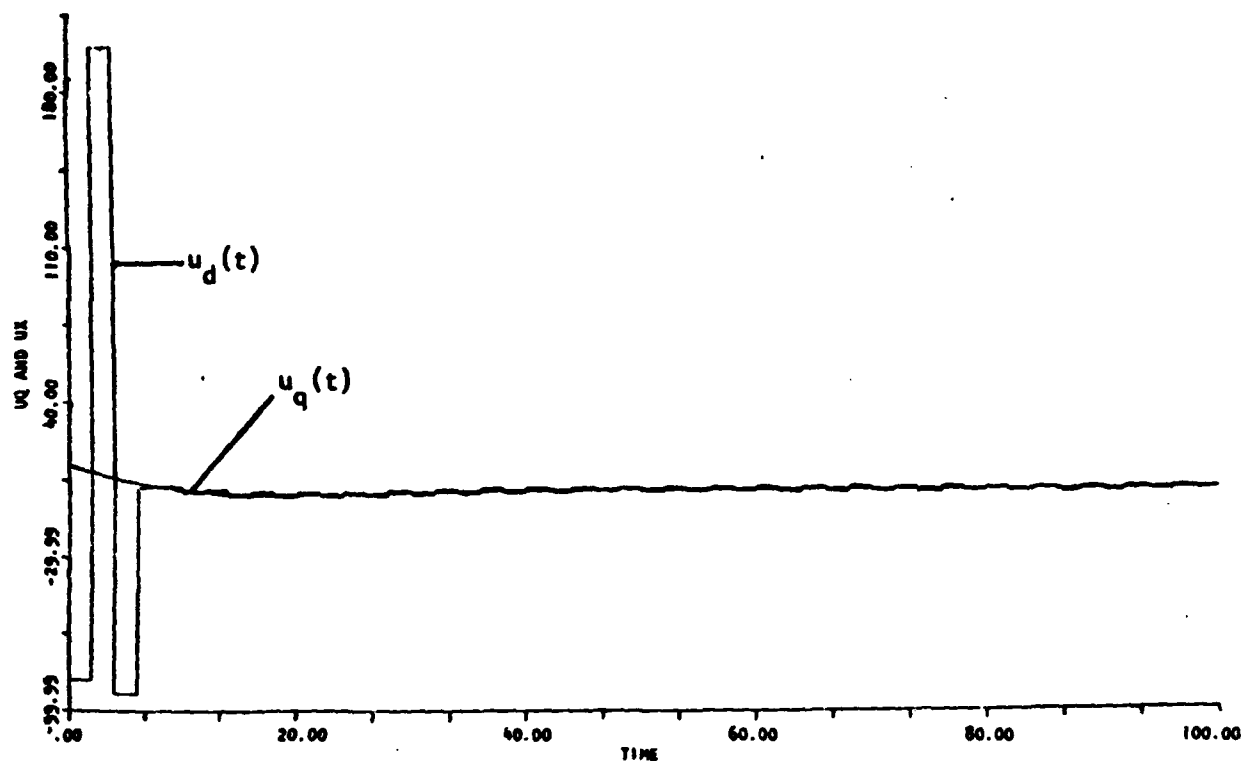
Figure 11. Control functions, Example 2; $q_2(t)$ and $x_2(t)$ matched, $x_2(t)$ unobservable. Continuous control $u_q(t)$ given by (16), digital control $u_d(t)$ defined through (13) and (19).
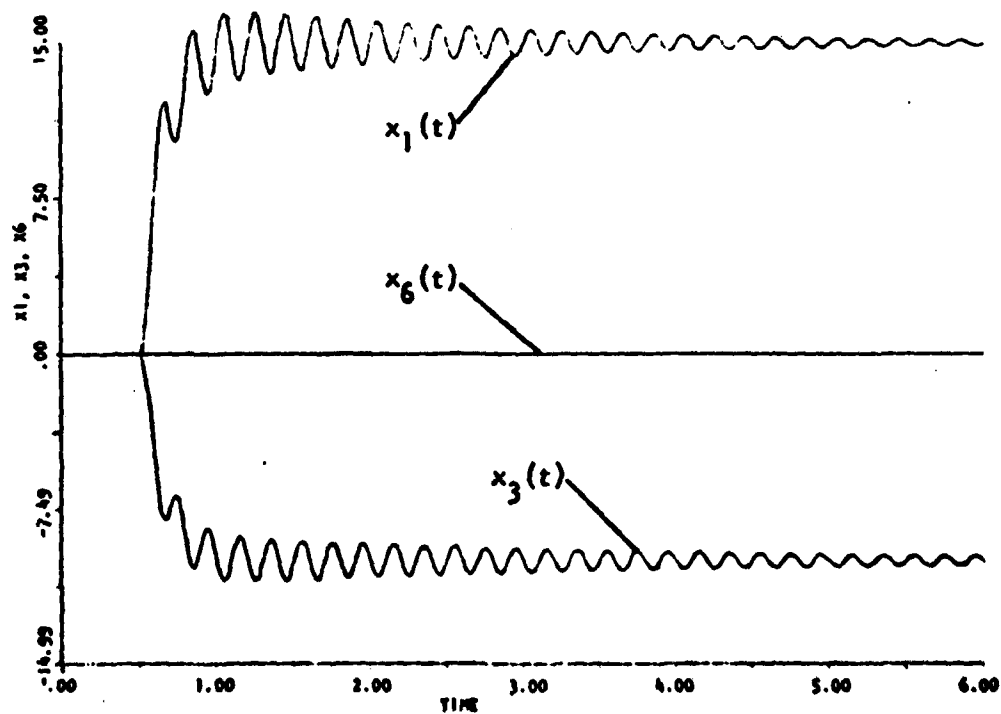

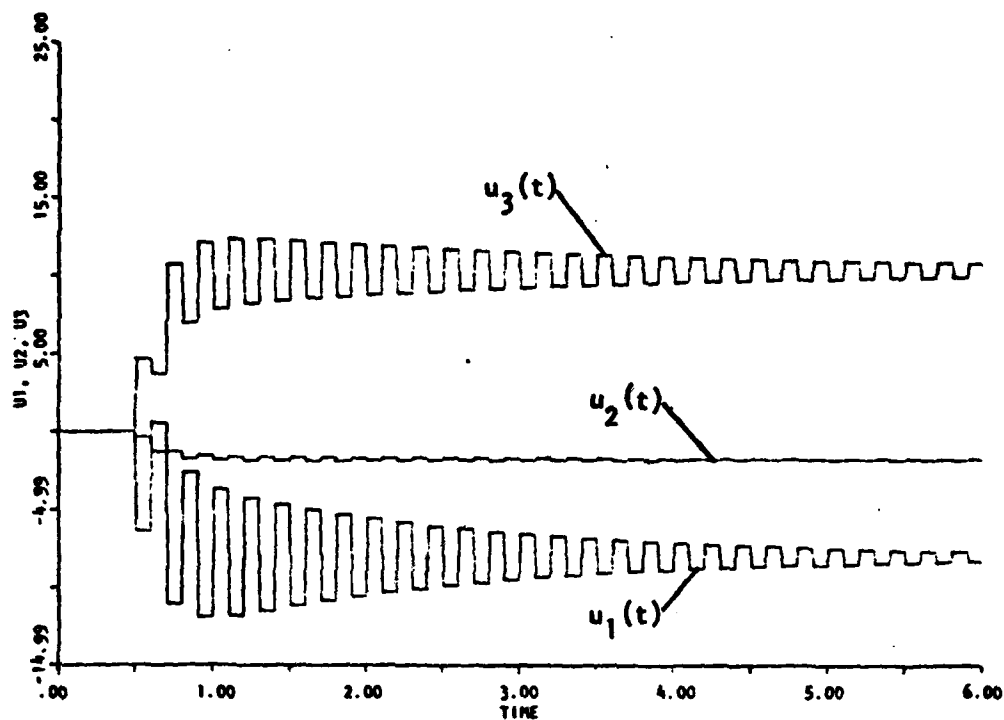Figure 12. $x_1(t)$, $x_3(t)$, and $x_6(t)$, Example 3.

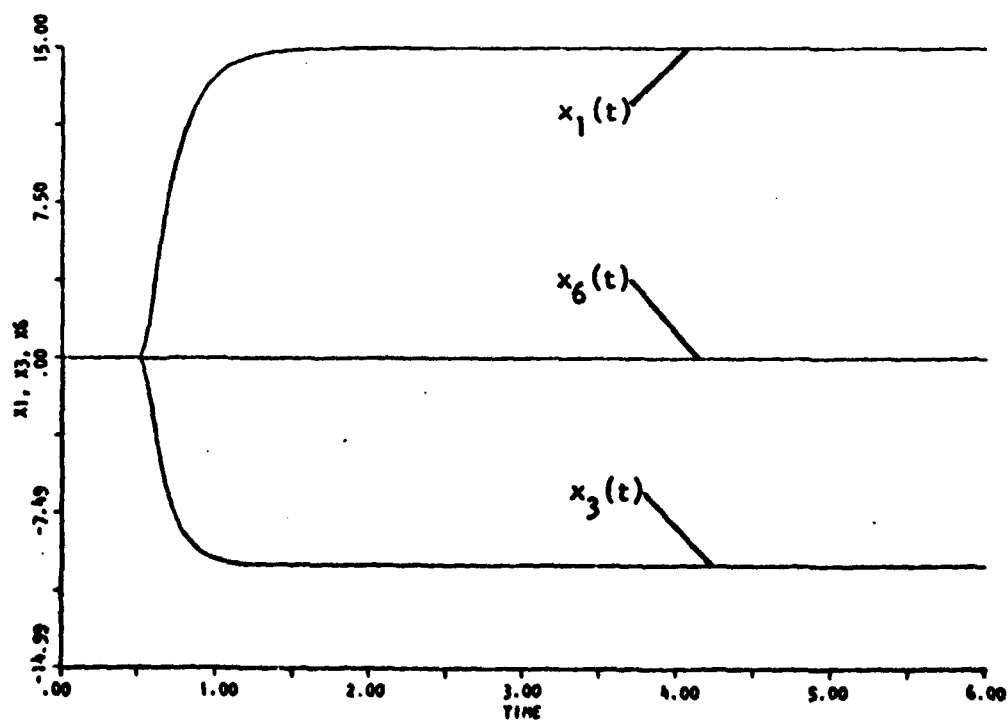Figure 13. $u_1(t)$, $u_2(t)$, and $u_3(t)$, Example 3.

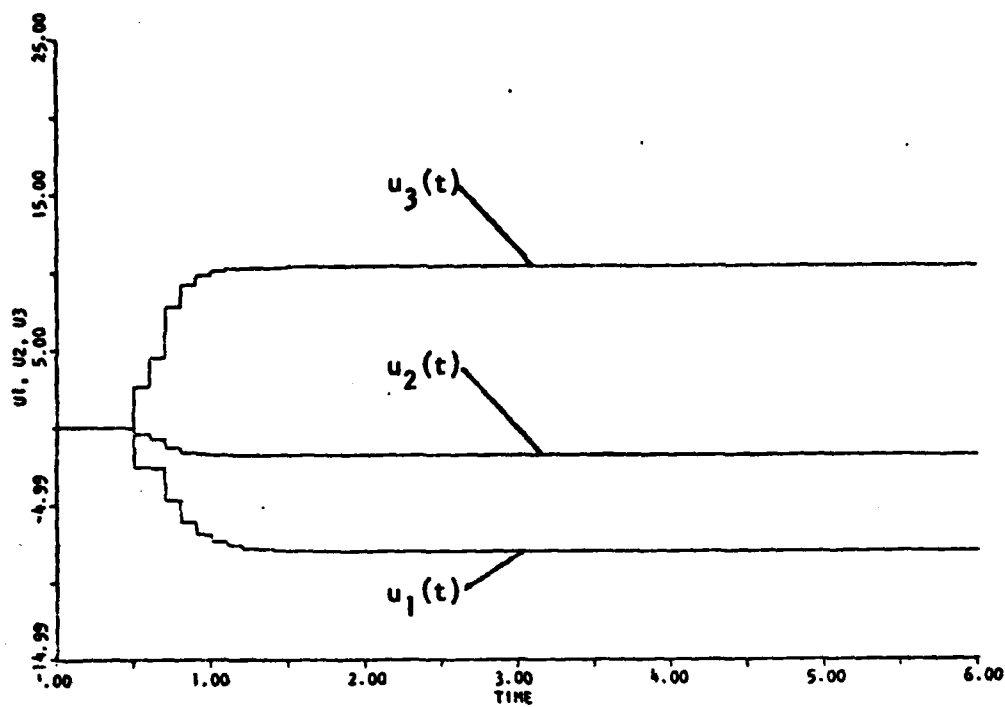Figure 14. $x_1(t)$, $x_3(t)$, and $x_6(t)$, Example 3; control smoothing approach.

Figure 15. $u_1(t)$, $u_2(t)$, and $u_3(t)$, Example 3; control smoothing approach.